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2003 J. Phys. A: Math. Gen. 36 8817

(http://iopscience.iop.org/0305-4470/36/33/307)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 8817–8827 PII: S0305-4470(03)63231-2

Travelling wave solutions and proper solutions to the two-dimensional Burgers–Korteweg–de Vries equation*

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Received 9 May 2003, in final form 1 July 2003
Published 5 August 2003
Online at stacks.iop.org/JPhysA/36/8817

Abstract

In this paper, we study the two-dimensional Burgers–Korteweg–de Vries (2D-BKdV) equation by analysing an equivalent two-dimensional autonomous system, which indicates that under some particular conditions, the 2D-BKdV equation has a unique bounded travelling wave solution. Then by using a direct method, a travelling solitary wave solution to the 2D-BKdV equation is expressed explicitly, which appears to be more efficient than the existing methods proposed in the literature. At the end of the paper, the asymptotic behaviour of the proper solutions of the 2D-BKdV equation is established by applying the qualitative theory of differential equations.

PACS numbers: 02.30.Jr, 02.30.Oz, 04.20.Jb

1. Introduction

The last few decades have seen an enormous growth in the applicability of nonlinear models and in the development of related nonlinear concepts. This has been driven by modern computer power as well as by the discovery of new mathematical techniques, which include two contrasting themes: (i) the theory of dynamical systems, most popularly associated with the study of chaos, and (ii) the theory of integrable systems associated, among other things, with the study of solitons. However, not all systems arising from physical phenomena are integrable, for example, the two-dimensional Burgers–Korteweg–de Vries (2D-BKdV) equation. Therefore, a direct method together with qualitative analysis for treating such nonlinear systems appears to be more powerful and important. Applications of nonlinear models range from atmospheric science to condensed matter physics and to biology, from the smallest scales of theoretical particle physics up to the largest scales of cosmic structure.

* Main results have been presented at 'The 2003 Texas Partial Differential Equations Conference', University of North Texas, Denton, USA.

Consider the 2D-BKdV equation

$$(U_t + \alpha U U_x + \beta U_{xx} + s U_{xxx})_x + \gamma U_{yy} = 0 \tag{1}$$

where α, β, s and γ are real constants and $\alpha\beta s\gamma \neq 0$. Equation (1) is a two-dimensional generalization of the Burgers–Korteweg–de Vries equation

$$U_t + \alpha U U_x + \beta U_{xx} + s U_{xxx} = 0 \tag{2}$$

which arises from many different physical contexts as a nonlinear model equation incorporating the effects of dispersion, dissipation and nonlinearity. Johnson derived (2) as the governing equation for waves propagating in a liquid-filled elastic tube [1] and Wijngaarden and Gao used it as a nonlinear model in the flow of liquids containing gas bubbles [2] and turbulence [3]. Grad and Hu used a steady state version of (2) to describe a weak shock profile in plasmas [4].

During the last few decades, many theoretical issues concerning the exact solutions of 2D-BKdV equation have received considerable attention. Barrera and Brugarino applied Lie group analysis to study the similarity solutions of (1) and examined some features of these invariant solutions, but the explicit travelling wave solution to (1) was not shown [5]. Li and Wang used the Hopf–Cole transformation and a computer algebra system to study (1) and found an exact travelling wave solution to (1) [6]. In the mean time, Ma proposed a bounded travelling wave solution to (1) by applying a special solution of square Hopf–Cole type to an ordinary differential equation [7]. These two methods were compared to each other, and the solutions are proved to be equivalent by Parkes [8]. Fan obtained the same result by using an extended tanh-function method for constructing multiple travelling wave solutions of nonlinear partial differential equations in a unified way [9]. Recently, Fan *et al* [10] claimed that a new complex line soliton for the 2D-BKdV equation was obtained by making use of the same technique as described in [9], and Elwakil *et al* [11] claimed that a new travelling solitary wave solution was obtained by using a modified extended tanh-function method. In our recent papers [12–14], we studied equation (1) by utilizing the first integral method and the Painlevé analysis, respectively, and obtained a more general travelling wave solution in terms of elliptic functions.

In the present paper, our purpose is to apply the qualitative theory of differential equations to the studies of travelling wave solutions and proper solutions of the 2D-BKdV equation. A travelling wave solution is obtained more efficiently by a direct method and the asymptotic behaviour of proper solutions is presented by using Hardy's theorem.

Assume that equation (1) has an exact solution in the form

$$U(x, y, t) = U(\xi) \qquad \xi = hx + ly - wt \tag{3}$$

where h, l, w are real constants to be determined. Substitution of (3) into equation (1) yields

$$-whU_{\xi\xi} + \alpha h^2 (UU_\xi)_\xi + \beta h^3 U_{\xi\xi\xi} + sh^4 U_{\xi\xi\xi\xi} + \gamma l^2 U_{\xi\xi} = 0.$$

Integrating the above equation twice with respect to ξ, then we have

$$sh^4 U_{\xi\xi} + \beta h^3 U_\xi + \frac{\alpha}{2} h^2 U^2 + \gamma l^2 U - whU = R$$

where R is the second integration constant and the first one is taken to be zero. Rewrite this second-order ordinary differential equation as

$$U''(\xi) - rU'(\xi) - aU^2(\xi) - bU(\xi) - d = 0 \tag{4}$$

where $r = -\frac{\beta}{sh}$, $a = -\frac{\alpha}{2sh^2}$, $b = \frac{wh-\gamma l^2}{sh^4}$ and $d = \frac{R}{sh^4}$.

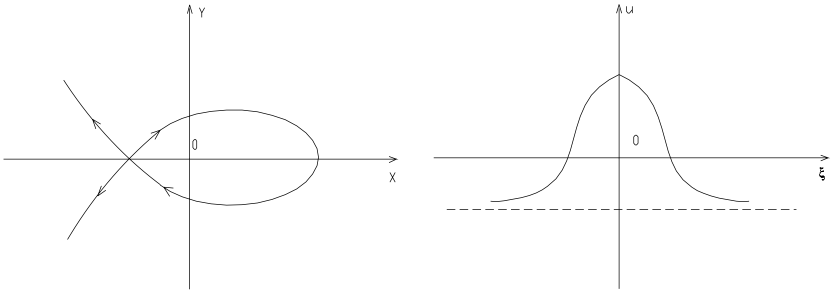

Figure 1. An isolated and closed orbit with an equilibrium point in the Poincaré phase plane represents a bell solitary wave solution in the (ξ, u)-plane.

When $\sqrt{b^2 - 4ad} > 0$, to remove the constant term in equation (4), let

$$U = -\frac{1}{a}u(\xi) - \frac{b}{2a} - \frac{\sqrt{b^2 - 4ad}}{2a} \tag{5}$$

and substituting (5) into equation (4) then yields

$$u''(\xi) + \delta u'(\xi) + u^2(\xi) - \mu u(\xi) = 0 \tag{6}$$

where $\delta = -r$ and $\mu = -\sqrt{b^2 - 4ad}$. Letting $v = u_\xi$, equation (6) is equivalent to

$$\begin{cases} \dot{u} = v = P(u, v) \\ \dot{v} = -\delta v - u^2 + \mu u = Q(u, v). \end{cases} \tag{7}$$

Equation (7) is a two-dimensional autonomous system. It is well known that plane autonomous systems are particularly useful in physics and engineering. The equilibrium point in the Poincaré phase plane always corresponds to a static state. If (u_0, v_0) is an equilibrium point of (7), then any orbit except itself cannot approach (u_0, v_0) within finite time. Conversely, if an orbit of (7) approaches (u_0, v_0) as $\xi \to \infty$ (or $-\infty$), then (u_0, v_0) must be an equilibrium point of (7). In the Poincaré phase plane, an isolated and closed orbit which has no equilibrium point on itself represents a periodic oscillation to equation (6) in the (ξ, u)-plane. An orbit which emanates from an equilibrium point and terminates at a different equilibrium point as $\xi \to \infty$ (or $-\infty$) represents a kink-profile solitary wave solution in the (ξ, u)-plane to equation (6). An isolated and closed orbit which emanates from an equilibrium point and also terminates at the same equilibrium point as $\xi \to \infty$ (or $-\infty$), represents a bell-profile solitary wave solution to equation (6) in the (ξ, u)-plane (see figure 1).

The rest of the paper is organized as follows. In section 2, we analyse the stability and bifurcation of system (7), which is equivalent to the 2D-BKdV equation (1) after making travelling wave transformation and integration. In section 3, a travelling wave solution to the 2D-BKdV equation (1) is obtained in terms of the Weierstrass elliptic function, and comparison with the existing results is shown at the end of this section. In section 4, the asymptotic behaviour of proper solutions of equation (1) is illustrated. Section 5 is a brief conclusion. We point out that some other nonlinear differential equations, such as Fisher's equation can be handled in a similar manner.

2. The stability of system (7)

Assume that $\delta < 0$ (the discussion for the case $\delta > 0$ is closely similar). System (7) has two equilibrium points $A(0, 0)$ and $B(u, 0)$.

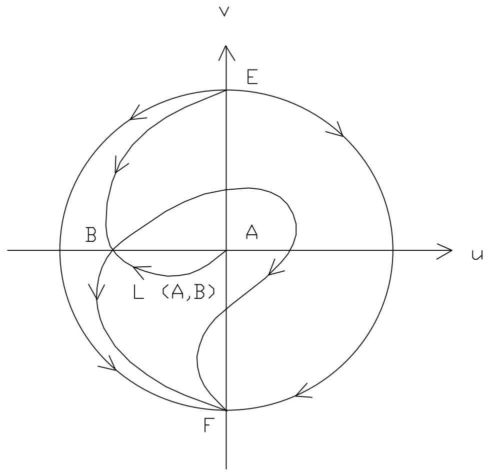

Figure 2. The global structure of system (7).

(i) $\frac{\delta^2}{4} > \mu > 0$, $A(0, 0)$ is a saddle point and $B(\mu, 0)$ is an unstable nodal point.
(ii) $-\frac{\delta^2}{4} < \mu < 0$, $A(0, 0)$ is an unstable nodal point and $B(\mu, 0)$ is a saddle point.
(iii) $\frac{\delta^2}{4} < \mu$, $A(0, 0)$ is a saddle point and $B(\mu, 0)$ is an unstable spiral point.
(iv) $-\frac{\delta^2}{4} > \mu$, $A(0, 0)$ is an unstable spiral point and $B(\mu, 0)$ is a saddle point.

Note that μ in (7) is negative, so we only need to consider the cases of (ii) and (iv).

Utilizing the Poincaré transformations

$$u = \frac{1}{z} \qquad v = \frac{X}{z} \qquad \mathrm{d}\tau = \frac{\mathrm{d}t}{z} \quad (z \neq 0)$$

and

$$u = \frac{Y}{z} \qquad v = \frac{1}{z} \qquad \mathrm{d}\tau = \frac{\mathrm{d}t}{z} \quad (z \neq 0)$$

one can find that (7) has two infinite equilibrium points E and F in v-axis. E is a source point and F is a sink point. The global structure of system (7) in the case (ii) is illustrated in figure 2.

From figure 2, one can see that except the equilibrium points A, B and the orbit $L(A, B)$, all other orbits in the Poincaré phase plane either depart from the infinite equilibrium point E or approach the infinite equilibrium point F as $\xi \to +\infty$. This implies that the v-coordinate of each point lying on the orbits except A, B and $L(A, B)$ is unbounded. By virtue of the mean-value theorem and the formula $\frac{\mathrm{d}v}{\mathrm{d}u} = -\delta - \frac{u^2-\mu u}{v}$, one can see that the u-coordinate of each point lying on the same orbits must be unbounded too.

Since $\frac{\partial P(u,v)}{\partial u} + \frac{\partial Q(u,v)}{\partial v} = -\delta \neq 0$, (7) has no closed orbit in the Poincaré phase plane according to the Poincaré–Bendixson theorem ([15]). This implies that, equation (6) neither has bell solitary wave solution, nor has bounded periodic travelling wave solution.

Since the plane autonomous system (7) is equivalent to equation (6), each nontrivial bounded travelling wave solution $u = u(\xi)$ of equation (6) in the (ξ, u)-plane corresponds to an orbit of system (7) in the Poincaré phase plane, in which the u-coordinate of each point is bounded. According to the above analysis, the unique orbit which satisfies this requirement under the condition $-\frac{\delta^2}{4} < \mu < 0$ is $L(A, B)$. Namely, in this case, the nontrivial bounded travelling wave solution $u(\xi)$ of equation (6) is unique. Moreover, utilizing the qualitative analysis as described in [12], we can conclude that when $r^2 > 4\sqrt{b^2 - 4ad}$, this nontrivial bounded travelling wave solution $u(\xi)$ of equation (6) is strictly monotone decreasing or increasing with respect to ξ, which depends on the sign of a.

Now, we are considering the stability near the origin for system (7). It will help us understand that the stability of system (7) actually depends on the sign of μ. Making the similarity transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -\delta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}. \tag{8}$$

and substituting (8) into (7), then yields

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\delta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \frac{\mu}{\delta} \cdot \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{\delta} \cdot \begin{pmatrix} -(x+y)^2 \\ (x+y)^2 \end{pmatrix}. \tag{9}$$

Fix $\delta > 0$ and consider the corresponding extension system

$$\begin{cases} \dot{x} = \frac{\mu}{\delta}(x+y) - \frac{1}{\delta}(x+y)^2 \\ \dot{\mu} = 0 \\ \dot{y} = -\delta y - \frac{\mu}{\delta}(x+y) + \frac{1}{\delta}(x+y)^2. \end{cases} \tag{10}$$

By virtue of the stable manifold theorem for a hyperbolic equilibrium point, system (10) has a two-dimensional centre manifold, which is tangent to the (x, μ)-plane at the equilibrium point $(x, \mu, y) = (0, 0, 0)$. Denote by $y = h(x, \mu)$ the centre manifold of (10) near the origin, where $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and satisfies

$$h(0,0) = 0 \qquad \frac{\partial h(x,\mu)}{\partial x} = \left.\frac{\partial h(x,\mu)}{\partial \mu}\right|_{(0,0)} = 0.$$

By making use of the Taylor expansion, $y = h(x, \mu)$ can be expressed as

$$h(x,\mu) = a_1 x^2 + b_1 x\mu + c_1 \mu^2 + o(3) \tag{11}$$

where $o(3)$ contains terms of power 3 or higher. Thus, we have

$$\begin{aligned} \dot{y} &= \frac{\partial h(x,\mu)}{\partial x} \cdot \dot{x} + \frac{\partial h(x,\mu)}{\partial \mu} \cdot \dot{\mu} \\ &= -\delta h(x,\mu) - \frac{\mu}{\delta}[x + h(x,\mu)] + \frac{1}{\delta}[x + h(x,\mu)]^2. \end{aligned} \tag{12}$$

Substituting (11) and the third equation of (10) into (12), we get

$$\begin{aligned} (2a_1 x + b_1\mu)&\left[\frac{\mu}{\delta}(x + a_1x^2 + bx\mu + c\mu^2 + o(3)\right] - \frac{1}{\delta}[x + a_1x^2 + bx\mu + c\mu^2 + o(3)]^2 \\ &= -\delta[a_1x^2 + bx\mu + c\mu^2 + o(3)] - \frac{\mu}{\delta}[x + a_1x^2 + bx\mu + c\mu^2 + o(3)] \\ &\quad + \frac{1}{\delta}[x + a_1x^2 + bx\mu + c\mu^2 + o(3)]^2. \end{aligned} \tag{13}$$

Equating the coefficients of x^2, $x\mu$ and μ^2 on both sides of (13), respectively, then yields

$$a_1 = \frac{1}{\delta^2} \qquad b_1 = -\frac{1}{\delta^2} \qquad c_1 = 0.$$

Returning to the x-equation in (10), we have

$$\dot{x} = \frac{1}{\delta}\left(1 - \frac{\mu}{\delta^2}\right)x(\mu - x) + o(3) \qquad (\dot{\mu} = 0). \tag{14}$$

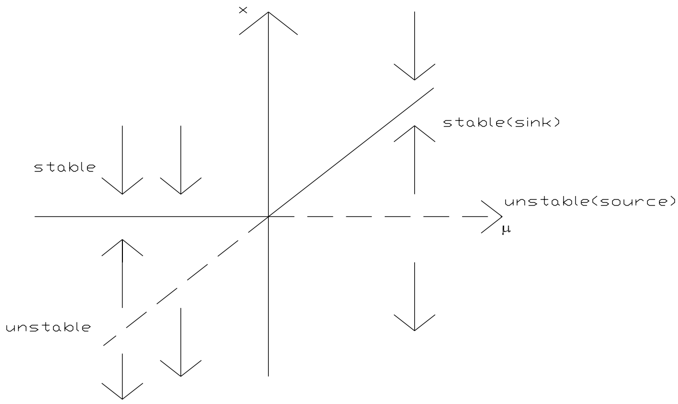

Figure 3. Transcritical bifurcation.

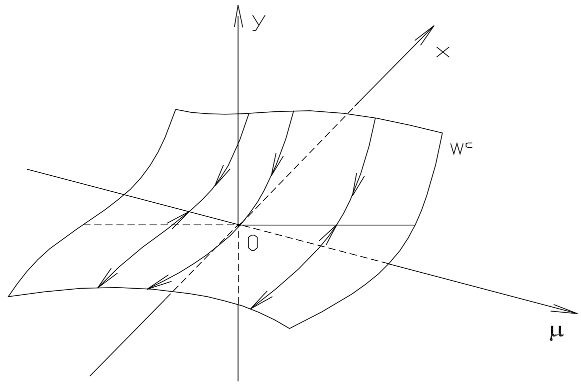

Figure 4. The centre manifold of system (10).

The equilibria of (14) are $x = 0$ and $x = \mu$. When δ and $1 - \frac{\mu}{\delta^2}$ have the same sign, the stable branches are illustrated in figure 3 by heavy lines and the unstable branches are indicated by broken lines. The trivial branch $x = 0$ loses stability at the bifurcation point $(x, \mu) = (0, 0)$. Simultaneously, there is an exchange of stability to the other branch. This implies that (7) loses its stability at $(u, \mu) = (0, 0)$. When $\delta > 0$ and $\mu < 0$ ($|\mu|$ is small), that is, when $\beta sh > 0$ and $(wh - \gamma l^2)^2 + 2\alpha Rh^2 > 0$, there exists a unique nontrivial bounded travelling wave solution to equation (1), which is stable and approaches to two bounded limits as $\xi \to +\infty$ and $\xi \to -\infty$, respectively. These two limits depend on the corresponding u-coordinates of equilibrium points A and B.

The centre manifold W^c of system (10) is depicted in figure 4. One can see that near the equilibrium point $(x, \mu, y) = (0, 0, 0)$, for each fixed μ, the stability of equilibrium points of system (9) is also determined by (14).

3. The travelling wave solution to the 2D-BKdV equation

In the preceding section, we concluded that under some particular conditions, equation (1) does have nontrivial bounded travelling wave solution which is monotonic. In this section, we restrict our attention to the study of seeking the exact travelling wave solution of equation (1) by a direct method.

First, make the natural logarithm transformation

$$\xi = -\frac{1}{\delta} \ln \tau$$

then equation (6) becomes

$$\delta^2 \tau^2 \frac{\mathrm{d}^2 u}{\mathrm{d}\tau^2} + u^2 - \mu u = 0. \tag{15}$$

Take the variable transformation as described in [16]

$$q = \tau^k \qquad u = \tau^{-\frac{1}{2}(k-1)} \cdot H(q)$$

then equation (15) becomes

$$\frac{\mathrm{d}^2 H}{\mathrm{d}q^2} = -\frac{1}{\delta^2 k^2} q^{\frac{1-5k}{2k}} H^2 \tag{16}$$

where $k = \sqrt{1 - \frac{4\sqrt{b^2-4ad}}{\delta^2}}$.

To simplify the coefficient on the right-hand side of (16) to be 1, we assume

$$\phi = q \qquad H = -\delta^2 k^2 \rho$$

then equation (16) reduces to

$$\frac{\mathrm{d}^2 \rho}{\mathrm{d}\phi^2} = \phi^m \rho^2 \tag{17}$$

where $m = \frac{1-5k}{2k}$.

Therefore, from equation (17) we can derive the following results immediately:

(I) When $k = \frac{1}{5}$, i.e., $\sqrt{b^2 - 4ad} = \frac{6r^2}{25}$, changing to our original variables, we obtain an exact solution to equation (1)

$$U(x, y, t) = -\frac{2\beta^2}{25\alpha s} \cdot \mathrm{e}^{-\frac{2\beta}{5sh}(hx+ly-wt)} \cdot \rho\left(\mathrm{e}^{-\frac{\beta}{5sh}(hx+ly-wt)} + c\right) + \frac{6\beta^2}{25\alpha s} + \frac{wh - \gamma l^2}{\alpha h^2} \tag{18}$$

where c is the arbitrary integration constant and $\rho(\phi, g_2, g_3)$ is the Weierstrass elliptic function with invariants g_2 and g_3 [17, 18] satisfying

$$\frac{\mathrm{d}^2 \rho}{\mathrm{d}\phi^2} = \rho^2. \tag{19}$$

Since $\rho(\phi) = 6(\phi + C_0)^{-2}$ is a particular solution of equation (19), apparently we can obtain a particular travelling solitary wave solution to equation (1) from (18) directly

$$U(x, y, t) = -\frac{12\beta^2}{25\alpha s} \cdot \frac{\mathrm{e}^{-\frac{2\beta}{5sh}(hx+ly-wt)}}{\left[\mathrm{e}^{-\frac{\beta}{5sh}(hx+ly-wt)} + C_1\right]^2} + \frac{wh - \gamma l^2}{\alpha h^2} + \frac{6\beta^2}{25\alpha s} \tag{20}$$

where C_1 is the arbitrary constant.

(II) In general, equation (17) has a nontrivial solution in the polynomial form $\rho(\phi) = c_0 \phi^\omega$, where

$$\omega = -m - 2 \qquad c_0 = (m + 2)(m + 3). \tag{21}$$

Reverting to system (7), we can see that this solution corresponds to the equilibrium point $B(\mu, 0)$ and the trivial solution $\rho = 0$ corresponds to the equilibrium point $A(0, 0)$.

Compared with various methods proposed for the 2D-BKdV equation (1) [6–14], the direct method introduced herein is more straightforward and less calculative. It is notable that (18) is not only more general than those solutions presented in [6–12], but also confirms the qualitative analysis discussed in the last section. Note that the solution derived by Ma [7, p L19] is only identical to (20) while c is positive. The case where c is negative is not discussed in [7]. By using the Hopf–Cole transformation and a computer algebra system, Li and Wang obtained a travelling wave solution to (1) in [6] simultaneously, which is proved to be equivalent to that obtained in [7] by Parkes in 1994 (see [8]). In [12], we applied the first integral method to study equation (1) and obtained a travelling solitary wave solution of the form (20), but (18) was not derived at that moment. In [13], we developed this technique to obtain (18). In our last paper [14], we obtained the same result by making use of the Painlevé analysis. However, the derivations for (18) in [13, 14] indeed contain tedious and complicated computations. As pointed out in [13], recently, a new complex line soliton claimed by Fan *et al* [10, p 378] and a new travelling solitary wave solution claimed by Elwakil *et al* [11, formula (12), p 183] are the particular cases of (20) where $C_1 = i$ and $C_1 = -1$, respectively.

4. The asymptotic behaviour of proper solutions

In section 3, we reduce the 2D-BKdV equation (1) to (17). For a certain value of m, it is possible to reduce (17) to a nonlinear equation with constant coefficients and thereby the way to the application of the Poincaré–Liapounov theory to the study of (17). Nevertheless, by applying the qualitative theory of differential equations, the solutions to equation (17), in general, cannot be expressed explicitly. Therefore, analysing the asymptotic behaviour of the solutions of equation (1) becomes sufficiently important and necessary. In order to isolate the large class tractable solutions, we employ the concept of proper solution, which is one that is real and nontrivial with continuous derivative for $\xi > \xi_0$. Since the arithmetic nature of m in equation (17) will have considerable influence upon the possible types of proper solutions, we only consider the positive proper solutions of equation (17). The arguments for the negative case can be handled similarly.

To present our discussion in a straightforward manner, we need the following theorem:

Hardy's theorem. Any solution of the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{P(u,t)}{Q(u,t)}$$

continuous for $t > t_0$, is ultimately monotonic, together with all its derivatives, and satisfies one or the other of the relations

$$u \sim At^j\,\mathrm{e}^{P(t)} \qquad u \sim At^j(\log t)^{1/l}$$

where $P(t)$ is a polynomial in t, A is constant and l is an integer.

Applying Hardy's theorem, we can obtain the asymptotic behaviour of proper solutions of the 2D-BKdV equation. That is, when $\sqrt{b^2 - 4ad} < \delta^2$, i.e., $\sqrt{\frac{(wh-\gamma l^2)^2+2\alpha Rh^2}{s^2h^4}} < \frac{\beta^2}{4s^2}$, proper solutions of the 2D-BKdV equation have the asymptotic form as follows:

$$U(x,y,t) \sim -\frac{\mathrm{sgn(s)}}{\alpha h^2}\sqrt{(wh-\gamma l^2)^2 + 2\alpha Rh^2} + \frac{wh-\gamma l^2}{\alpha h^2}. \tag{22}$$

Now, we prove (22). From equation (17), it is easy to see that $\rho(\phi)$ must be eventually monotone. Since if there is a point ϕ_0 such that $\rho'(\phi_0) = 0$, $\rho(\phi)$ can only have a minimum at $\phi = \phi_0$ due to the fact that $\rho'' = \phi^m\rho^2 > 0$. Hence ρ is eventually monotone decreasing or monotone increasing.

Now let us set $\rho = c_0\phi^{\omega}T$, here c_0 and ω have the same values as given in (21). The equation for T is

$$T'' + (2\omega - 1)T' + \omega(\omega - 1)(T - T^2) = 0. \tag{23}$$

Note that $T = 0$ and $T = 1$ are two trivial solutions to equation (23). Since $0 < k < 1$, we get $\omega = \frac{1}{2} - \frac{1}{2k} < 0$ and $2\omega - 1 < 0 < \omega(\omega - 1)$.

Consider the possible alternatives for T; we already know $T > 0$. In the case of the solutions in the region $0 < T < 1$. From the ultimate monotonicity of the solutions, we have $T \to 0$ or $T \to 1$ as $\phi \to \infty$. One can easily rule out the possibility that $T \to 0$. The characteristic roots of the linearization of (23) are given by $-\omega$ and $1 - \omega$. Since both roots are positive, it follows that $T = 0$ is a thoroughly unstable solution and thus that no other solution of (23) can tend to this as $\phi \to \infty$. Thus the alternative is $T \to 1$, which implies

$$\rho \sim \left(\frac{1}{2k} - \frac{1}{2}\right)\left(\frac{1}{2k} + \frac{1}{2}\right)\phi^{\frac{1}{2}-\frac{1}{2k}}. \tag{24}$$

In the case that T crosses $T = 1$, it must continue monotonically increasing, since any turning point must be a minimum. That T approaches a finite limit L greater than 1 is possible. In this case, $T' \to 0$ and $T'' \to 0$, so any finite limit including L must be a root of $T - T^2 = 0$. This yields a contradiction. Thus, we can deduce that $T \to \infty$. We now investigate this possibility by using Hardy's theorem. Setting $F = T'$, equation (23) reduces to

$$F\frac{\mathrm{d}F}{\mathrm{d}T} + (2\omega - 1)F + \omega(\omega - 1) = 0. \tag{25}$$

As $T \to \infty$, we have either

$$F \sim \mathrm{e}^{h(T)}T^{c_1} \tag{26}$$

where $h(T)$ is a polynomial in T, or

$$F \sim T^{c_2}(\log T)^{c_3} \tag{27}$$

where c_i $(i = 1, 2, 3)$ are constants.

Combine (25) with (26) or (27), respectively. Evaluation of the constants indicates that both cases lead to $F \geqslant T^{1+\epsilon}$ with $\epsilon > 0$ as $T \to \infty$. Going back to our assumption $F = \frac{\mathrm{d}T}{\mathrm{d}\phi}$, one can see that this is impossible if we are considering proper solutions to equation (23). Hence again if $T > 1$, we have $T \to 1$ as $\phi \to \infty$, which yields (24).

Using the inverse of transformations described in section 3, we have

$$H \sim -\delta^2k^2\rho(q) \sim -r^2 \cdot \frac{1 - k^2}{4} \cdot q^{\frac{1}{2}-\frac{1}{2k}}$$

and

$$u \sim \tau^{-\frac{1}{2}(k-1)} \cdot H(q) \sim -\sqrt{b^2 - 4ad}.$$

Making use of (5) and changing to the original variables, we obtain

$$\begin{aligned} U(x, y, t) &\sim \frac{\sqrt{b^2 - 4ad}}{2a} - \frac{b}{2a} \\ &\sim -\frac{\mathrm{sgn(s)}}{\alpha h^2}\sqrt{(wh - \gamma l^2)^2 + 2\alpha R h^2} + \frac{wh - \gamma l^2}{\alpha h^2}. \end{aligned}$$

This is the asymptotic behaviour of proper solutions of the 2D-BKdV equation (1).

5. Conclusion

In summary, in this work, first we analyse the stability and bifurcation of system (7), which is equivalent to the 2D-BKdV equation. Then an exact solution to the 2D-BKdV equation is expressed in terms of the Weierstrass elliptic function, and compared with the existing results as shown in section 3. Finally, the use of Hardy's theorem illustrates the asymptotic behaviour of proper solutions of equation (1). The technique for seeking travelling wave solutions described herein appears to be more efficient and less computational than those methods used in [6–14]. It is worthwhile to point out that the above results do not depend on the particular example of the 2D-BKdV equation. One can definitely apply the coordinate transformations given in section 3 and Hardy's theorem to study many nonlinear differential equations.

Some representative equations are listed below:

(1) Fisher's equation [19]: $u_t = vu_{xx} + su(1-u)$.
(2) Modified Burgers–KdV equation [20]: $u_t + \beta u^2 u_x + \mu u_{xx} - su_{xxx} = 0$.
(3) Compound KdV equation [20]: $u_t + \alpha uu_x + \beta u^2 u_x + su_{xxx} = 0$.
(4) Generalized Klein–Gordon equation [21, 22]: $u_{tt} - (u_{xx} + u_{yy}) + \alpha^2 u_t + g(uu^*)u = 0$.
(5) Nonlinear Schrödinger equation [21, 22]: $iu_t + u_{xx} - u_{yy} + g(uu^*)u = 0$.
(6) Emden equation [16]: $vu'' + 2u' + \alpha v^m u^n = 0, \alpha > 0$.
(7) Emden–Fowler equation [16]: $\frac{\mathrm{d}}{\mathrm{d}t}\left(t^q \frac{\mathrm{d}u}{\mathrm{d}t}\right) \pm t^\delta u^n = 0$.
(8) An approximate sine-Gordon equation [23]: $u_{tt} + r_1 u_t - r_2 \Delta u_{(n)} + u - \frac{1}{6}u^3 = d_3 u^3 + d_2 u^2 + d_1 u + d_0$.
(9) Combined dissipative double-dispersive equation [24]: $u_{tt} - \alpha_1 u_{xx} - \alpha_2 u_{xxt} - \alpha_3 (u)^2_{xx} - \alpha_4 u_{xxxx} + \alpha_5 u_{xxtt} = 0$.

Acknowledgments

The author sincerely thanks Professor Goong Chen for his kind help, Professor Ciprian Foias for fruitful discussions and one anonymous referee for his/her precious and constructive comments.

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